From Atiyah to Lurie.

20 years of TQFT.

Outline:
0. Introduction.
1. Category \( n \text{-Cob} \) and TQFT.
2. Frobenius algebras and 2D TQFT.
3. Perspective: Extended TQFT, Lurie Theorem.

Diagram:
- Space of states \( \Sigma_0 \), \( \Sigma_1 \), \( \Sigma_f \).
- Spacetime \( \rightarrow \) (evolution).
- Space of states.
- \( M \) \( n \)-dimensional manifold.
- Time evolution operator.
- \( \Sigma_0, \Sigma_1 \) \((n-1)\)-dimensional manifolds.
- \( \downarrow \) \( \mathbb{Z} \).
- \( \downarrow \) \( \rightarrow \) \( \mathbb{Z}(M) \) linear map.

\( V_0, V_1 \) vector spaces + axioms.
Axioms:
- Topological \( M \cong M' \Rightarrow Z(M) = Z(M') \)
- Quantum \( \Sigma_0 = \Sigma' \cup \Sigma'' \Rightarrow V_0 = V'_0 \otimes V''_0 \)
  \( Z(\phi_{n-1}) = \text{Id} \)
  \( Z(\otimes): V'_0 \otimes V''_0 \xrightarrow{\text{sym}} V''_0 \otimes V'_0 \)

Def: An \( n \)-dimensional \( TQFT \) is a symmetric monoidal functor:
\[
(\text{nCob}, \sqcup, \phi_{n-1}, \tau) \xrightarrow{Z} (\text{Vect}_k, \otimes, \text{Id}, \sigma).
\]

Q: But... what does it mean?

Some applications:
1. Invariants of closed manifolds:

\[
\begin{array}{c}
\phi \\
\downarrow \quad Z \\
\text{Id} \\
\end{array}
\]

So \( Z \) associates a scalar \( \text{Id} \in k \) to \( M \)

- 2 -
(2) Linear representations of orientation preserving diffeomorphism groups of an \( n \)-dimensional closed manifold.

Let \( f: M \to M \) be such diffeomorphism. Then \( M \times [0,1] \xrightarrow{\text{Id} \times f} M \) defines a morphism 
\[
C(f)M \to M \quad \text{in} \quad (n+1)\text{Cob}.
\]
So its image \( f \) is an element in \( \text{End}(Z(M)) \).

More formally:

**Def**: A category \( C \) consists of the following data:

1. a class of objects \( \text{Ob}(C) : A, B, C, \ldots \)
2. a class of morphisms \( \text{Mor}(C) : f: A \to B, g: C \to D, \ldots \)
3. every morphism \( f \) has a specified domain \( \text{dom}(f) \) and codomain \( \text{cod}(f) \), which are objects of \( C \).
4. for any pair \( f, g \) of morphisms such that \( \text{cod}(f) = \text{dom}(g) \) there exists a morphism \( g \circ f \) which is called a composition of \( g \) and \( f \).
5. for any object \( A \) there exists an identity morphism \( 1_A : A \to A \).

This data is required to satisfy the following conditions:

6. for any composable triple of morphisms \( h, g, f \) the equality \( h \circ (g \circ f) = (h \circ g) \circ f \) holds.
7. for any morphism \( f: A \to B \) the equations 
\[
f \circ 1_A = f = 1_B \circ f
\]
Examples: Sets, Top, Vect_k

Q: Is there a category of ... categories?

Def: A functor $F: C \to D$ is a pair of mappings $F: \text{ob}(C) \to \text{ob}(D)$ and $F: \text{mor}(C) \to \text{mor}(D)$ such that:

1. for any morphism $f$ in $C$:
   \[ F(\text{dom}(f)) = \text{dom}(F(f)), \]
   \[ F(\text{cod}(f)) = \text{cod}(F(f)). \]

2. for any composable pair of morphisms $f, g$:
   \[ F(g \circ f) = F(g) \circ F(f). \]

3. for any object $A$ in $C$:
   \[ F(1_A) = 1_{F(A)} \]

Examples:
- $U: \text{Grp} \to \text{Sets}$ forgetful functor
- $F: \text{Sets} \to \text{Grp}$ free group functor
- $\pi_1: \text{Top}^\ast \to \text{Grp}$ fundamental group functor

The category $\text{nCob}$

Objects: closed oriented $(n-1)$-manifolds
Morphisms: diffeomorphism classes of $n$-cobordisms
Def: A cobordism from $\Sigma_0$ to $\Sigma_1$ is a manifold $M$ with boundary and maps

$$\partial^\text{in} M \subseteq M \supseteq \partial^\text{out} M$$

$\Sigma_0 \xrightarrow{\text{diffeo}} \Sigma_1$

Examples:

- $\text{ is a cobordism } S^1 \sqcup S^1 \to S^1$
- $\text{ is a cobordism } \phi_1 \to \phi_1$
- $\text{ is a cobordism } S^1 \sqcup S^1 \sqcup S^1 \to S^1 \sqcup S^1$

Identity morphism for $\Sigma$:

$$\Sigma \to \Sigma \xrightarrow{\text{in}} \Sigma \xrightarrow{\text{out}} \Sigma \times [0,1]$$

Composition:

$$\Sigma_0 \to \Sigma_1 \to \Sigma_2$$

glueing i.e.

pushout in $\text{Top}$
This defines the category $n\text{Cob}$.

**Def.** A strict monoidal category $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ equipped with a functor:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

$$(A, B) \mapsto A \otimes B$$

and an object $I \in \mathcal{C}$ such that:

1. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
2. $I \otimes A = A = A \otimes I$.

**Examples:**

- $(\text{Vect}_{\text{lk}}, \otimes, \text{lk})$, where $\text{lk} \otimes V = V$ (!)
- $(n\text{Cob}, \Pi, \phi_{n-1})$

**Def.** A symmetry on a monoidal category is a family of maps

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

such that:

1. $A \otimes B \xrightarrow{1_{A \otimes B}} A \otimes B$

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A
\end{array}$$

2. $A \xrightarrow{c} C$

$$\begin{array}{ccc}
A & \xrightarrow{\sigma_{A,C}} & C
\end{array}$$
3. \[ A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \]

\[ f \circ g \]

\[ A' \otimes B' \xrightarrow{\sigma_{A',B'}} B' \otimes A' \]

Exercise: check that \( \Sigma, \Sigma' \) defines a symmetry on \( \text{nCob} \)

\[ \Sigma \quad \Sigma' \]

\[ \Sigma' \quad \Sigma \]

Definition (M. Atiyah, 1988)

A Topological Quantum Field Theory in dimension \( n \) is a symmetric monoidal functor

\[ \mathbb{Z} : \left( \text{nCob}, \cup, \phi_{n,1}, \tau \right) \rightarrow \left( \text{Vect}_k, \otimes, k, \sigma \right) \]

Goal: present \( 2\text{Cob} \) by generators and equations.

Proposition. \( (2\text{Cob}, \cup, \tau) \) is generated by:

\[ \times, \circ, \bigcirc, \bigtriangleup, \bigtriangleup \]

Proof: Observe that connected oriented surfaces are classified by: 1. genus, 2. number of in-boundary components, 3. number of out-boundary components.
**Proposition** The relations are "topologically evident."

**Examples:**

\[
\begin{align*}
\includegraphics[width=0.3\textwidth]{example1} & = \includegraphics[width=0.3\textwidth]{example2} \\
\includegraphics[width=0.3\textwidth]{example3} & = \includegraphics[width=0.3\textwidth]{example4} \\
\end{align*}
\]

... 

\[
\begin{align*}
\includegraphics[width=0.3\textwidth]{example5} & = \includegraphics[width=0.3\textwidth]{example6} \\
\end{align*}
\]

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**Def:** A **Frobenius k-algebra** is a \( k \)-vector space with:

- multiplication \( \mu: A \otimes A \to A \)
- and unit \( \eta: k \to A \)
- comultiplication \( \delta: A \to A \otimes A \)
- and counit \( \varepsilon: A \to k \)

such that:

\[
\begin{align*}
\includegraphics[width=0.3\textwidth]{comultiplication}\quad & = \quad \includegraphics[width=0.3\textwidth]{counit} \\
\includegraphics[width=0.3\textwidth]{multiplication}\quad & = \quad \includegraphics[width=0.3\textwidth]{unit}
\end{align*}
\]

... and some more
Theorem.
\[ \text{Sym Mon Fun}((\text{2Cob}, \sqcup, \phi, \tau), (\text{Vect}_k, \otimes, k, \sigma)) \approx \text{cFA} \]

Proof:
\[
\begin{array}{c}
\mathbb{Z} \longrightarrow \mathbb{Z}(S^1) \\
\mathbb{Z} \longleftarrow A
\end{array}
\]

s. th.
\[ \mathbb{Z}(S^1) = A \]

+ conditions.

Theorem
\[ \text{Sym Mon Fun}((\text{2Cob}, \sqcup, \phi, \tau), (C, \otimes, I, \sigma)) \approx \]
\[ \approx \text{commutative Frobenius objects in } C. \]

Def: An **Extended TQFT** is a symmetric monoidal \((\infty, 1)\)-functor from the symmetric monoidal \((\infty, n)\)-category \(n\text{Cob}\) to the symmetric monoidal \((\infty, n)\)-category \(\text{Chain}(k)\).
2-category: objects \\
morphisms \\
2-morphisms

+ axioms

Examples: a monoidal category with one object
- the 2-category $\text{Cat}$
- $X$ topological space; $\pi_2 X$ fundamental 2-groupoid
  - objects = points of $X$
  - morphisms = paths in $X$
  - 2-morphisms = homotopies between paths

$(\infty, n)$-category $\mathfrak{n}\text{Cob}$:
  - objects: oriented 0-manifolds (sets of points)
  - morphisms: 1-dimensional cobordisms
  - 2-morphisms: 2-dimensional cobordisms
  - $n$-morphisms: $n$-dimensional
  - $(n+1)$-morphisms: diffeomorphisms
  - $(n+2)$-morphisms: isotopies

Theorem (Lurie, 2008)
$\text{Sym Mon Fun}_{(\infty, n)}(\mathfrak{n}\text{Cob}, C)^\sim \{\text{fully dualisable})
\{\text{objects in } C\}$

\[ \mathbb{Z} \rightarrow \mathbb{Z} (+) \]