Interacting Frobenius Algebras and the Structure of Multipartite Entanglement

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Introduction

- For many quantum algorithms and protocols, entangled states are where the magic happens
- Often the study of entanglement is concerned with numeric measures, or exhaustive (and exhausting!) classification theorems
- While this is useful, we take a courser-grained approach, looking at *behavioural*, or *algebraic* properties of entangled states
- We show that a small number of interacting algebraic structures can describe a large class of multipartite entangled states
- The term "algebraic" is massively overloaded, so lets clarify

Entanglement and Algebra

- First note, we shall switch freely between the notion of a vector $|\psi\rangle \in H$ and its associated map

$$\psi : \mathbb{C} \to H :: 1 \mapsto |\psi\rangle$$

- Information flows *across* entangled states. Taking this seriously, we can regard states $|\Psi\rangle \in A \otimes B$ as maps

$$\widetilde{\Psi} : A^* \to B :: \langle a| \mapsto (\langle a| \otimes 1)|\Psi\rangle$$

Entanglement and Algebra

- If we fix an isomorphism $\phi : A \cong A^*$, then $\widetilde{\Psi}\phi$ is a new map:

$$\widehat{\Psi} : A \to B$$

- Consider a tripartite state $|\Psi\rangle \in A \otimes A \otimes A$, then we can do this trick to get

$$\widehat{\Psi} : A \otimes A \to A$$

- $\widehat{\Psi}$ is a bilinear multiplication of vectors in A. If we impose conditions like associativity, etc. $(A, \widehat{\Psi})$ becomes an *algebra* over A

LOCC and SLOCC

- We'll consider pure state entanglement on qubits
- States $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent up to LOCC (local operations and classical communication) iff there exist unitaries $U_i : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$|\Psi\rangle = (U_1 \otimes \ldots \otimes U_n)\,|\Phi\rangle$$

- $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent up to stochastic LOCC, or SLOCC, if they can be interconverted, as above, but with non-zero probability
- This is equivalent to saying that for local *invertible* maps L_i

$$|\Psi\rangle = (L_1 \otimes \ldots \otimes L_n)\,|\Phi\rangle$$

Entanglement, LOCC, and SLOCC

- There are two SLOCC classes in $\mathbb{C}^2 \otimes \mathbb{C}^2$, $|00\rangle$ and $|00\rangle + |11\rangle$
- In 3 qubits, there are only two genuine (non-product) states, up to SLOCC:

$$|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle \qquad |\mathrm{W}\rangle = |100\rangle + |010\rangle + |001\rangle$$

Associative Algebra

- $(A, *)$ is an *associative k-algebra* if A is a k-vector space and $(- * -) : A \times A \to A$ is an associative, bilinear multiplication defined on A
- It is *unital* if a vector $e \in A$ acts as a unit
- Since $*$ is bilinear, it factors uniquely through the tensor product

$$\left(A \times A \xrightarrow{*} A\right) = \left(A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\mu} A\right)$$

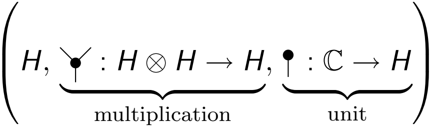
- From now one, will shall focus on unital algebras on finite-dimensional, complex Hilbert spaces

Coalgebras

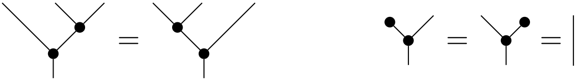
- So, we can alternately define a unital algebra $(H, *, |e\rangle)$ as a triple (H, μ, η), where $\eta :: 1 \mapsto |e\rangle$ and
  - μ as associative: $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$
  - η is the unit: $\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1$
- We could also define an algebra on the dual space $(H^*, \delta^*, \epsilon^*)$, or, equivalently, a *coalgebra* on H
- A (unital) *coalgebra* is a triple $(H, \delta : A \longrightarrow A \otimes A, \epsilon : \mathbb{C} \longrightarrow H)$ that is
  - *coassociative*: $(1 \otimes \delta)\delta = (\delta \otimes 1)\delta$
  - *counital*: $(\epsilon \otimes 1)\delta = (1 \otimes \epsilon)\delta = 1$

Associative Algebra (Graphically)

- An associative algebra is a triple



- Such that



And upside down...

- A coassociative coalgebra is a triple

(H,  $: H \to H \otimes H$ (comultiplication),  $: H \to \mathbb{C}$ (counit))

- Such that

Commutative Frobenius Algebras, Definition 1

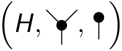
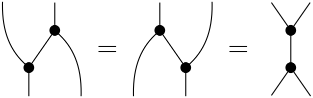
- Since we've fixed in isomorphism with the dual space, any algebra on H cannonically yields a coalgebra. Equivalently, we could start with one and make some axioms
- A *commutative Frobenius algebra* is a quintuple:

$$(H, \mu, \eta, \delta, \epsilon)$$

  - (H, μ, η) is a commutative algebra,
  - (H, δ, ϵ) is a cocommutative coalgebra, and
  - $(\mu \otimes 1)(1 \otimes \delta) = (1 \otimes \mu)(\delta \otimes 1) = \delta\mu$

Commutative Frobenius Algebras, Definition 1

- A *commutative Frobenius algebra* is a quintuple:

- Such that
  -  is an commutative algebra,
  -  is a cocommutative coalgebra, and
  - 

Example

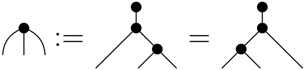
- For a finite-dimensional Hilbert space H, fix an ONB $\{|i\rangle\}$
- Let $\delta :: |i\rangle \mapsto |ii\rangle$, copy each of the basis vectors
- $\epsilon :: |i\rangle \mapsto 1$ uniformly deletes them
- $\mu :: |ii\rangle \mapsto |i\rangle, |ij\rangle \mapsto 0$ fuses them (aka, it performs the Schur product w.r.t. the basis)
- $\eta :: 1 \mapsto \sum |i\rangle$ is the unit of μ

Finding the Tripartite States

- We started by taking about tripartite states. What's that have to do with Frobenius algebras?
- A tripartite state $|\Psi\rangle \in H \otimes H \otimes H$ is a map

$$\Psi : \mathbb{C} \to H \otimes H \otimes H$$

- ...so is this:



- We can talk about the tripartite states *generated* by Frobenius algebras
- We're particularly interested in *symmetric* states, hence *commutative* Frobenius algebras

Commutative Frobenius Algebras, Definition 2

- A third (and final) definition...
- A *commutative Frobenius algebra* is a triple

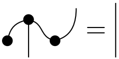
$$(\Psi \in H \otimes H \otimes H, \phi : H \otimes H \to \mathbb{C}, \epsilon : H \to \mathbb{C})$$

- Such that
  - $(1 \otimes \phi \otimes 1)(\Psi \otimes \Psi)$ is symmetric, and
  - $\psi := (\epsilon \otimes 1 \otimes 1)\Psi$ generates an inverse to ϕ, or a "cap". I.e.:

$$(1 \otimes \phi)(\psi \otimes 1) = 1$$

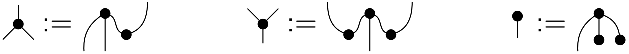
Commutative Frobenius Algebras, Definition 2

- A *commutative Frobenius algebra* is a triple

- Such that
  -  is symmetric, and
  - 

Equivalence of Definitions

- It is well known that (def 1) $\iff$ (def 2)
- If we let:



then (def 2) $\implies$ (def 1)

- The other way, let:

then (def 1) $\implies$ (def 2)

Examples

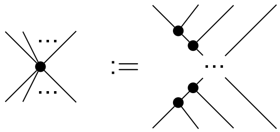
- Writing the Frobenius algebra as a triple ( ,  , ), we see some familiar faces...

- $$\left(\underbrace{|000\rangle + |111\rangle}_{\text{GHZ state}}, \underbrace{\langle 00| + \langle 11|}_{\text{Bell state}}, \langle +|\right)$$

- $$\left(\underbrace{|100\rangle + |010\rangle + |001\rangle}_{\text{W state}}, \underbrace{\langle 10| + \langle 01|}_{\text{EPR state}}, \langle 0|\right)$$

Normal Forms

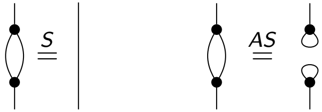
- These equivalent definitions are reflective of the fact that Frobenius algebras are largely topological in nature
- For "cannonical trees," we use the following notation:



- We call this thing a *spider*
- In fact, any connected, *acyclic* graph can be rewritten as such a tree. The equivalences on the previous slide are all special cases of this.
- ...and *any* graph can be rewritten as a tree with some loops in the middle

Special and Anti-special

- The only thing left to determine is what happens at the loops
- There's a lot of choices, but these two are particularly interesting:

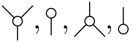
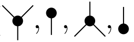


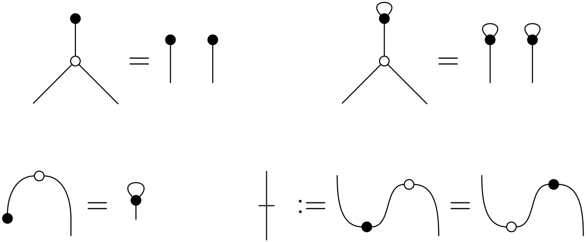
- If a CFA satisfies the left property, we call it *special*
- If it satisfies the right, we call it *anti-special*

The Punchline

- Why do we like these? Two classification results:
  - **Theorem:** A commutative Frobenius algebra over $\mathbb{C}^2$ is locally convertable to a special one *if and only if* its induced tripartite state is SLOCC with GHZ.
  - **Theorem:** A commutative Frobenius algebra over $\mathbb{C}^2$ is locally convertable to an anti-special one a *if and only if* its induced tripartite state is SLOCC with W.
- Since all such states are geniune tripartite, and GHZ and W are the only tripartite qubit states, we have an easy correlary:
  - **Corrolary:** Every CFA over $\mathbb{C}^2$ is either locally equivalent to a special CFA or to an anti-special CFA.

Interaction Rules

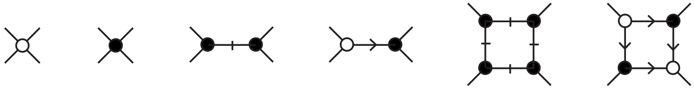
- Why should you care?
- If we denote the GHZ algebra by () and the W algebra by (), we notice some interaction properties:



- And many more. Some we don't know yet!

Four-Partite and Beyond

- Looking at these interaction behaviours, we found representatives for lots of 4-partite SLOCC classes:



- As well as came up with an inductive technique for cooking them up...

Inductive Classification

- In 2006,L. Lamata, J. Leon, D. Salgado, E. Solano published a paper called *Inductive classification of multipartite entanglement under SLOCC*, where they identitfied an inductive technique for classifying multipartite systems
- They did this by reguarding a state in $(\mathbb{C}^2)^{\otimes n}$ as a map from $(\mathbb{C}^2)^{\otimes(n-1)}$ to $\mathbb{C}^2$, and looking at the vectors spanning the right singular subspace $\mathfrak{W}$
- Since these vectors are in $(\mathbb{C}^2)^{\otimes(n-1)}$, they apply the inductive procudure down to a base case of $n = 2$, where the SLOCC classes are trivially computable

Inductive Classification

- At $n \geq 4$, there are infinitely many SLOCC classes, so Lamata and others introduce free parameters into the classification scheme, identifying SLOCC "superclasses"
- The representatives of these n partite classes are all of the form $|0, \Psi_0\rangle + |1, \Psi_1\rangle$. So, to represent this graphically, we would like to have some kind of map:

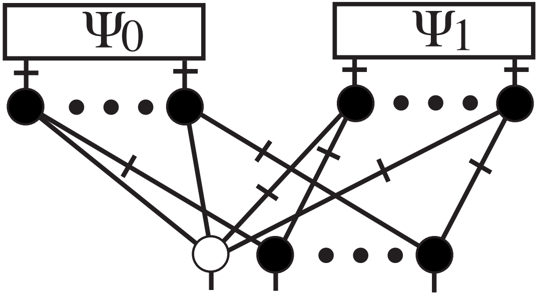
$$|\Psi\rangle_0 \otimes |\Psi\rangle_1 \longmapsto |0, \Psi_0\rangle + |1, \Phi_1\rangle$$

- But thats impossible, because it's not linear. However, this is SLOCC, so we only care up to a scalar
- We can do almost as good by just projecting out the unwanted state:

$$|\Psi\rangle \otimes |\Phi\rangle \longmapsto \langle p|\Phi\rangle|0, \Psi\rangle + \langle p|\Psi\rangle|1, \Phi\rangle$$

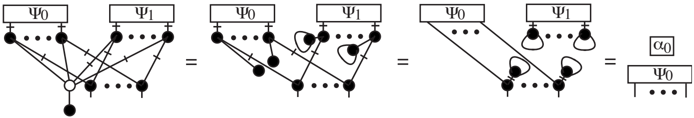
Inductive Classification

- We do this graphically like so:



Inductive Classification

- (,) is a basis for $(\mathbb{C}^2)^*$, so we use them to project the control bit:



- ...and similarly, projecting by yields $\alpha_1|\Psi_1\rangle$.

Conclusion

- Tripartite states define Frobenius algebras
- Since (up to SLOCC) there are two tripartite states over $\mathbb{C}^2$, there are two types of commutative Frobenius algebras, special and anti-special
- In studying the interactions between a special Frobenius algebra and its associated anti-special FA, a rich algebraic theory crops up, which we are only beginning to understand
- We can already capture many behavioural properties of states, and provide an inductive technique for "building to order" n-partite states

Future Work

- We'd like to do this in arbitrary monoidal categories, not just FHilb. Axiomatize!
- The red-green theory can actually be encoded as a GHZ/W pair. The details need to be worked out.
- Questions of exhaustiveness, efficiency, and universality
- When and how can we go backwards from a vector to a graph?

Thanks!

This is joint work with Bob Coecke. Big chunks also due to Ross Duncan, Bill Edwards, and Simon Pedrix.

Questions?